

## **SOLVING TWO-DIMENSIONAL NAVIER - STOKES EQUATION WITH BOUNDARY ELEMENT METHOD**

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### **ABSTRACT**

The boundary element method (BEM) has been recently recognized as one of the very useful tools in analyzing many of the mechanical problems. Not needing calculation field discretization and solving the problem using boundary information are the most important features and advantages of this method. Using this method for solving nonlinear problems and also linear problems dealing with integral expressions on the field resulted from the main expression has gained more importance with proposing the Dual Reciprocity Method (DRM). This paper discusses how this method is applied for solving incompressible Navier-Stokes equations, and the results show that this method has highly desired accuracy.

**KEYWORDS:** Boundary Element Method, Navier-Stokes, Dual Reciprocity Method, Fundamental Solution, Integral Equations

### **1.0 INTRODUCTION**

BEM is one of the most recent numerical solution methods for most mechanical problems. This method began around 1970 as a strongly practical tool and developed in the following years. The use of this method was first only limited to solving linear governing equations. From around 1980 onwards, using this method was proposed in papers for solving nonlinear equations for fluids. So far the method has been able to solve many of nonlinear problems under specific conditions and it is expected to be able to analyze many of the nonlinear problems more easily than finite difference and finite element methods in the near future. Not needing grid generation in calculation field, solving the problem only on the boundary points and actually solving the problem with one less dimension and thus needing less computer memory are the major advantages of this method. Regarding that only boundary discretization is done in this method, the available model building codes such as ARIES and PATRAN can be easily used for preprocessing. The difficulties of grid generation will be revealed more than ever, especially if the boundaries of the computational domain in several levels need to be changed for accessing the optimized design. One of the other privileges of this method is that after finding the solution on the boundaries, the solutions could be found with high accuracy on any intended point in the domain, and not only on specific grid points.

In the BEM, the integral equation equal to the governing equation is first found, which includes several integral expressions. If the governing equation is one that has no fundamental solution, the integral expressions achieved will all be boundary integrals. Otherwise, the governing equation should be divided into two parts, so that one part has an operator with the fundamental solution available for it, and the other part is considered to be a source expression. This will result into an integral expression on the computational domain, calculating the integral of which is the greatest difficulty in the

BEM. There are different methods for solving this problem, of which DRM is preferred by many of the researchers for general and common source expressions. This method is used in this paper for solving the problem. Using this method, there is no need for gridding in the calculation field and only some of the internal points should be used, the coordinates of which can be recognized easily.

This paper discusses solving the permanent incompressible Navier-Stokes equations in a cavity with the boundary element and dual reciprocity methods. The Navier-Stokes equations were first examined with the BEM by Bush and Tanner [1].

Using the fundamental solution for Stokes equations and considering the nonlinear expressions for convection as source expressions, these researchers found the boundary integral formulation for the equations, where in addition to the boundary integrals; they also faced an integral on the domain, and inevitably had to get help from domain discretization for solving it.

After that, a similar formulation was proposed by Tosaka and Onishi [2], using which the integral resulted from nonlinear expressions was first turned into two integrals by a part to part integration where one of the integrals was boundary and the other was an integral on the domain including the velocities and derivative of the fundamental solution. Although domain discretization was used again for solving this integral, but since this integral lacked the derivative expressions of velocities, it was more practical and preferred to the method proposed by Bush and Tanner. Also, using the Stokes fundamental solution, Dargush and Banerjee [3] analyzed the Navier-Stokes equations along with the equation of energy for permanent, incompressible and thermo viscous flow using the BEM, in which they too used field discretization for calculating the integral on the domain.

The penalty function methods common to finite difference and element formulations were also used by Kuroki et al. [4] and Kitagawa and Brebbia [5] for solving Navier-Stokes equations using the BEM. In this method, the penalty function is used for linking the pressure to divergence in the velocity field, and therefore the Navier-Stokes equations turn to Navier elasticity with an expression of physical energy resulted from nonlinear convection expressions.

In addition to directly solving the permanent Navier-Stokes equations, the nonpermanent form of these equations was also used by many researchers for finding a permanent solution. Using time discretization, Tosaka and Onishi turned these equations into a nonlinear convection equation and Stokes linear equations. Kakuda and Tosaka [7] used two time-splitting methods which led to a convection-diffusion equation and a linear Euler equation. A review and comparison of the three methods mentioned above were also done by Tosaka and Kakuda [8]. Moreover, a solution to these equations using the fundamental solution for nonpermanent Stokes equations was performed by Dargush and Banerjee [9], which led to a boundary integral formulation and an integral expression on the time and domain.

In all the methods mentioned above, there is always an integral expression on the domain resulted from nonlinear convection expressions, which requires discretization of computational domain for being calculated. When discussing the discretization of computational domain, the popularity of the BEM is reduced, since field discretization is also added to it and it no longer considers only the discretization for solving the problem. This paper discusses solving two dimensional Navier-Stokes equations in a cavity using the dual reciprocity technique in the BEM, by which the integral expression on the computational domain will be turned into boundary integrals using a series of specific solutions [10, 11]. Thus, using this method, the nonlinear equations can be analyzed only by discretization of computational boundaries, without field

discretization being required.

## 2. GOVERNING EQUATIONS

Regarding that the fluid is assumed incompressible; the conservation of mass is defined by the continuity equation as follows:

$$\nabla \cdot \vec{u} = 0 \tag{1}$$

Problem-Solving Flowchart is shown in Figure 1

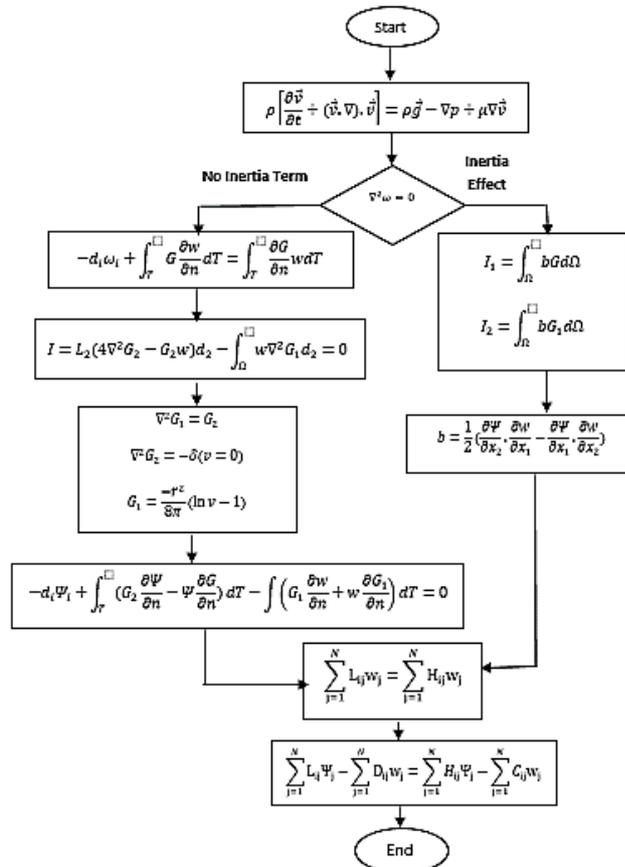


Figure 1: Flowchart of the Solving Using BEM

Regarding that the density and fluid velocity are constant and also there is no mass energy, the conservation of momentum equation will be:

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \mu \nabla^2 \vec{u} \tag{2}$$

Regarding these equations, the velocity-pressure formulation for Navier Stokes equations in the

$$u_{i,j} = 0 \tag{3}$$

$$\rho u_i u_{i,j} = -p_j + \mu(u_{i,j} + u_{j,i}) \tag{4}$$

And if  $g_i$  represents convection terms, we will have

$$\mu(u_{i,j} + u_{j,i}) - p_j = g_i \tag{5}$$

In fact, the nonlinear expressions are separated from linear expressions by  $g_i$ . The integral equation regarding the

partial differential equations system mentioned above for point  $x \in \Omega$  is [12].

$$\begin{aligned}
 u_k(x) &= \int_{\Gamma} K_{kj}(x, y) u_j(y) d\Gamma_y \\
 &- \int_{\Gamma} G_{ik}(x, y) t_i(y) d\Gamma_y \\
 &+ \int_{\Gamma} G_{ik}(x, y) g_i(y) d\Omega_y
 \end{aligned}
 \tag{6}$$

Where  $t_i(y) = \sigma_{ij}(u_i(y), p(y))n_j(y)$  the vector of traction energy  $\sigma_{ij}(u_i, p) = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$  is the tensor of the tension resulted from the flow field  $(u_i(y), p)$  and  $n_i(y)$  is the normal vector of the single regressive on the  $\Gamma$  boundary in the  $y$  point.  $G_{ik}(x, y)$ , called Stokes let is the single fundamental solution to Stokes equations, which is in position  $y$  and direction  $k$ .  $K_{kj}$  is also the traction vector resulted from this Stokes let, and is called the stress let. Figure 2 shows how the Stokes generates using concentrated load in different directions. The Green's function is defined as follows:

$$\begin{aligned}
 G_{ik}(x, y) &= -\frac{1}{4\pi\mu} \left( \frac{r_i r_k}{r^2} - \delta_{ik} \ln r \right) \\
 K_{kj}(x, y) &= -\frac{1}{\pi} \frac{r_i r_j r_k}{r^4} n_k(y) \\
 r &= |x - y|, \quad r_i = x_i - y_i
 \end{aligned}
 \tag{7}$$

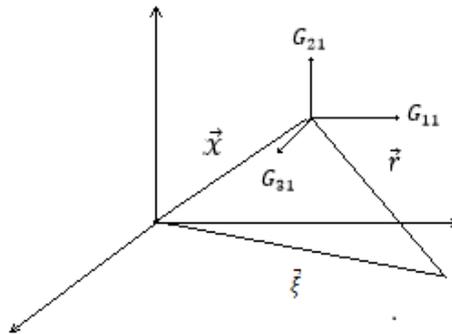


Figure 2: The Effect of the Concentrated Load in Point  $\xi$  on Point  $x$

Now if the interior point  $x$  is leaded to the boundary, considering the dual potential leap characterization, the integral equation for the points on the boundary will be as follows

$$\begin{aligned}
 c_{kj} u_j(x) &= \int_{\Gamma} K_{kj}(\zeta, y) u_j(y) d\Gamma_y \\
 &- \int_{\Gamma} G_{ik}(\zeta, y) t_i(y) d\Gamma_y \\
 &+ \int_{\Gamma} G_{ik}(\zeta, y) g_i(y) d\Omega_y
 \end{aligned}
 \tag{8}$$

Where  $c_{ij} = \Theta(\xi)\delta_{ij}/2\pi$  and  $\Theta(\xi)$  is the internal angle in  $\xi \in \Gamma$  point. As you can see, one of the integral expressions above is defined on the  $\Omega$  area, for calculating which the dual reciprocity method will be used.

### 2.1 The Dual Reciprocity Method (DRM)

In the dual reciprocity method, the  $g_i$  function in the  $f$  function spaces is expanded as follows

$$g_i = \rho u_j u_{i,j} = \sum_{m=1}^p f(x, y^m) \alpha_i^m \delta_d \quad (9)$$

The above series includes the  $f$  functions, which are depended to the intended geometry, and the  $\alpha$  indices. The way these indices are found will be explained later. With regard to what has been said, we can write:

$$\int_{\Omega} G_{ik}(x, y) g_i(y) d\Omega_y = \sum_{m=1}^p \alpha_i^m \int_{\Omega} G_{ik}(x, y) f(y, z^m) \delta_{il} d\Omega_y \quad (10)$$

Now assume that  $\check{u}_{il}$  and  $\check{t}_{il}$ , are solutions to the following divergent Stokes equation:

$$\mu \frac{\partial^2 \check{u}_{il}(x, y^m)}{\partial x_i \partial x_i} - \frac{\partial \check{p}_l(x, y^m)}{\partial x_i} = f(x, y^m) \delta_{il} \quad (11)$$

$$\frac{\partial \check{u}_{il}}{\partial x_i} = 0 \quad (12)$$

Now if Green's theorem is applied in this problem, we have

$$\begin{aligned} \hat{u}_d(x, z^m) &= \int_{\Gamma} K_{kj}(x, y) \hat{u}_{ji}(y, z^m) d\Gamma_y \\ &\quad - \int_{\Gamma} G_{ik}(x, y) \hat{t}_{il}(y, z^m) d\Gamma_y \\ &\quad - \int_{\Omega} G_{ik}(x, y) f(y, z^m) \delta_{il} d\Omega_y \end{aligned} \quad (13)$$

Or

$$\begin{aligned} \int_{\Omega} G_{ik}(x, y) f(y, z^m) \delta_{il} d\Omega_y &= \hat{u}_{il}(x, z^m) \\ &\quad - \int_{\Gamma} K_{kj}(x, y) \hat{u}_{ji}(y, z^m) d\Gamma_y \\ &\quad - \int_{\Gamma} G_{ik}(x, y) \hat{t}_{il}(y, z^m) d\Gamma_y \end{aligned} \quad (14)$$

Where  $\check{t}_{il}$  is the vector of traction energy for the above divergent Stokes equation and is as follows:

$$\hat{t}_{il}(y, z) = \sigma_{ij}(\hat{u}_{il}(y, z), \hat{p}_l(y, z)) n_j(y) \quad (15)$$

Finally, by placing (13) in (9) we have

$$\begin{aligned} &\int_{\Omega} G_{ik}(x, y) f(y, z^m) \delta_{il} d\Omega_y \\ &= \sum_{m=1}^p \alpha_i^m \{ \hat{u}_{kl}(x, z^m) \\ &\quad - \int_{\Gamma} K_{kj}(x, y) \hat{u}_{ji}(y, z^m) d\Gamma_y \end{aligned} \quad (16)$$

$$+ \int_{\Gamma} G_{ik}(x, y) \hat{t}_{il}(y, z^m) d\Gamma_y \}$$

And therefore, from (16) and (6), an integral relation defined only on the boundary will be resulted as follows

$$\begin{aligned} u_k(x) = & \int_{\Gamma} K_{kj}(x, y) u_j(y) d\Gamma_y \\ & - \int_{\Gamma} G_{ik}(x, y) t_j(y) d\Gamma_y \\ & + \sum_{m=1}^P \alpha_1^m \{ \hat{u}_{kl}(x, z^m) \\ & - \int_{\Gamma} K_{kj}(x, y) \hat{u}_{jl}(y, z^m) d\Gamma_y \\ & + \int_{\Gamma} G_{ik}(x, y) \hat{t}_{il}(y, z^m) d\Gamma_y \} \end{aligned} \quad (17)$$

And if  $\xi$  is one of the boundary points, regarding the dual potential leap on the boundary, the integral equation above will be as follows

$$\begin{aligned} c_{kj}(\xi) u_j(\xi) = & \int_{\Gamma} K_{kj}(\xi, y) u_j(y) d\Gamma_y \\ & - \int_{\Gamma} G_{ik}(\xi, y) t_j(y) d\Gamma_y \\ & + \sum_{m=1}^P \alpha_1^m \{ c_{kj}(\xi) \hat{u}_{jl}(\xi, z^m) \\ & - \int_{\Gamma} K_{kj}(\xi, y) \hat{u}_{jl}(y, z^m) d\Gamma_y \\ & + \int_{\Gamma} G_{ik}(\xi, y) \hat{t}_{il}(y, z^m) d\Gamma_y \} \end{aligned} \quad (18)$$

Now, the way of finding the  $\check{u}_{il}$  specific solution and the related traction vector,  $\check{t}_{il}$  and also  $\alpha_1$  indices will be examined. On one hand, the  $f$  function should be first determined for finding the specific solution regarding Equation (11). Therefore, the following items should be determined:

- The  $f$  function
- The specific solution
- The  $\alpha$  Coefficients

### 2.1.1 The f Function

Studying the results of previous researches in DRM show that although different f functions can be used as a base for expanding the source expression, but the best results are usually gained by choosing  $f=1+r$ , where  $r=|x-y|$ . Regarding the references available on DRM, choosing this kind of function was not based on mathematical analysis, but on the researchers' experiments. However, recent studies on the interpolation theory based on radial basis functions somehow justify this choice [13].

### 2.1.2 The Specific Solution

In order to find the specific solution regarding the divergent Stokes equation, the same method is used as the one used for the fundamental solution in the convergent Stokes equation [14]. First, the second place tensor  $\hat{u}_{ij}(x, y)$  is defined as follows based on an auxiliary potential  $\Psi(r)$ :

$$\hat{u}_{ij}(x, y) = \frac{\partial^2 \Psi}{\partial x_k \partial x_k} \delta_{ij} - \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \quad (19)$$

With a derivation from the relation above, it can be seen that the continuity equation is automatically satisfied. Placing (19) in (11) gives

$$\begin{aligned} \mu \left( \frac{\partial^4 \Psi}{\partial x_j \partial x_j \partial x_k \partial x_k} \delta_{ij} - \frac{\partial^4 \Psi}{\partial x_j \partial x_j \partial x_i \partial x_i} \right) \\ - \frac{\partial \hat{p}_1}{\partial x_i} = (1 + r) \delta_{ij} \end{aligned} \quad (20)$$

Now if we assume that  $\Psi(r)$  applies to the biharmonic equation, meaning

$$\mu \frac{\partial^4 \Psi}{\partial x_j \partial x_j \partial x_k \partial x_k} = 1 + r \quad (21)$$

Then the pressure field should satisfy the following equation:

$$\mu \frac{\partial^4 \Psi}{\partial x_j \partial x_j \partial x_i \partial x_i} + \frac{\partial \hat{p}_1}{\partial x_i} = 0 \quad (22)$$

And therefore, the following will be resulted:

$$\hat{p}_1 = -\mu \frac{\partial^3 \Psi}{\partial x_i \partial x_i \partial x_i} \quad (23)$$

The solution for equation (21) will be

$$\mu \Psi = \frac{r^4}{64} + \frac{r^5}{225} \quad (24)$$

Now with placing the  $\Psi$  potential gained in equations (19) and (23), the following relations are found for divergent Stokes flow field:

$$\hat{U}_{ij}(x, y) = \frac{1}{\mu} \left[ r^2 \left( \frac{3}{16} + \frac{4r}{45} \right) - \sigma_{ij} r_i r_j \left( \frac{1}{8} + \frac{r}{15} \right) \right] \quad (25)$$

$$\hat{p}_1(x, y) = -r_1 \left( \frac{1}{2} + \frac{r}{3} \right) \quad (26)$$

The tension tensor regarding this field and also the related traction vector will be

$$\sigma_{il} (\hat{U}_{ij}(x, y), p_1(x, y)) = r_1 \left(\frac{1}{4} + \frac{r}{5}\right) \delta_{il} + r_1 \left(\frac{1}{4} + \frac{r}{5}\right) \delta_{ill} + r_1 \left(\frac{1}{4} + \frac{r}{5}\right) \delta_{il} - \frac{2r_1 r_j r_l}{15r} \quad (27)$$

$$f_{ij}(x, y) = \sigma_{il} n_i = r_1 \left(\frac{1}{4} + \frac{r}{5}\right) n_i + r_1 \left(\frac{1}{4} + \frac{r}{5}\right) n_i + r_1 \left(\frac{1}{4} + \frac{r}{5}\right) n_i \delta_{il} - \frac{2r_1 r_j r_l}{15r} n_j \quad (28)$$

### 2.1.3 The $\alpha$ Indices

In order to find the  $\alpha$  indices, Equation (9) is calculated for all the nodes on the boundary and in the field. Therefore,

$$\alpha_i^k = [F(y^m, y^k)]^{-1} g_i(y^m) = [F(y^m, y^k)]^{-1} \rho u_{i,j}(y^m) \quad (29)$$

In order to be released from the derivative expression,  $u_i(x)$  is itself expanded again in the same space; meaning

$$u_i(x) = F(x, y^m) \beta_i^m \quad (30)$$

Thus it can be written that

$$\beta_i^k = [F(y^m, y^k)]^{-1} u_i(y^m) \quad (31)$$

Is derived from (30)

$$u_{i,j}(x) = \left[ \frac{\partial}{\partial x_j} F(x, y^m) \right] \beta_i^m \quad (32)$$

And by placing the  $\beta$  numbers from Equation (31), the following is resulted

$$u_{i,j}(x) = \left[ \frac{\partial}{\partial x_j} F(x, y^m) \right] \times [F(y^m, y^k)]^{-1} u_i(y^m) \quad (33)$$

Now if the matrix  $U_i(y^m, y^n)$  is defined as follows

$$U_i(y^m, y^n) = u_i(y^m) \delta_{mn} \quad (34)$$

It can be concluded that

$$[\rho u_{i,j}] (y^m) = \rho U_i(y^m, y^n) \left[ \frac{\partial}{\partial x_j} F(x, y^m) \right] [F(y^m, y^k)]^{-1} u_i(y^m) \quad (35)$$

Regarding the Equations (29) and (35), we have

$$\alpha_i^k = [F(y^m, y^k)]^{-1} \rho u_i(y^m, y^n) \left[ \frac{\partial}{\partial x_j} F(x, y^m) \right] \quad (36)$$

$$[F(y^m, y^k)]^{-1} u_i(y^m)$$

Considering that the Equation above depends on the solution  $U_i$ , the solution method will require repetition. In order to begin the solution, an answer is firstly considered as the initial assumption (usually Stokes field solution), and then solving the problem with the repetitious methods will be done until convergence to the solution.

### 2.2 Numerical Solution Method

Equation (18) is the basis of calculations for the numerical solution method. In the numerical method, the defined integrals on the boundary are written as the integration of the defined integrals on the boundary elements, and then, by

calculating these integrals with the Gauss method, an algebraic equation system is achieved [15, 16]. By solving this equation system and determining the answer in each level, the right part of the equations is corrected and these steps continue until the convergence of the solution. Regarding Equation (18), the following can be written:

$$\begin{aligned}
 c_{kj}(\xi)u_i(\xi) - \sum_{n=1}^N \int_{\Gamma_n} k_{kj}(\xi, y)u_j(y)d\Gamma_y \\
 + \sum_{n=1}^N \int_{\Gamma_n} G_{ik}(\xi, y)t_i(y)d\Gamma_y \\
 = \sum_{m=1}^P \alpha_l^m \{c_{kj}(\xi)\hat{u}_{jl}(\xi, z^m) \\
 - \sum_{n=1}^N \int_{\Gamma_n} k_{kj}(\xi, y)\hat{u}_{jl}(y, z^m)d\Gamma_y \\
 + \sum_{n=1}^N \int_{\Gamma_n} G_{ik}(\xi, y)\hat{t}_{il}(y, z^m)d\Gamma_y\}
 \end{aligned} \tag{37}$$

In this relation too, if  $\xi$  is an internal point,  $c_{kj}(\xi)$  equals  $\delta_{kj}$ , and if the point is boundary, it will be calculated regarding the material's angle in that point. Figure 3 shows the components of the  $r$  vector which are  $r_1$ ,  $r_2$  and  $r$  in the  $n_k$  direction which is shown by  $r_k$ .

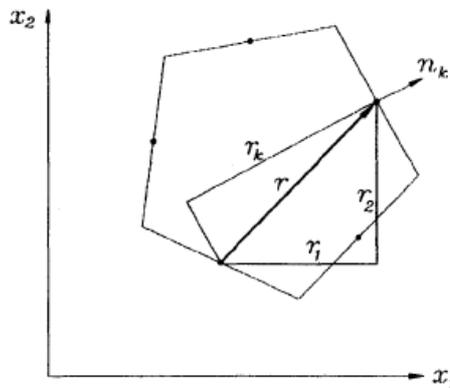


Figure 3: The  $r_1$ ,  $r_2$  and  $r$  Vectors in Relation to Each Other

When the integration is done on an element with the concentrated load, we will face a singular integral with weak technique. In this case, regarding figure 3, the following can be written

$$r_k = 0, \quad r_1 = r \cos \theta, \quad r_2 = r \sin \theta \tag{38}$$

Now if  $G_{11} = \int_{\Gamma_1} G_{11}(\xi, y)d\Gamma_y$ , with regard to (7), we will have

$$\begin{aligned}
 G_{11} &= \int_{\Gamma_1} \frac{1}{4\pi\mu} \left( \frac{r_1^2}{r^2} - \ln r \right) d\Gamma \\
 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4\pi\mu} (\cos^2\theta - \ln r) dr
 \end{aligned} \tag{39}$$

$$\begin{aligned}
&= \int_0^{\frac{1}{2}} \frac{1}{4\pi\mu} (\cos^2\theta - \ln r) dr \\
&= \frac{1}{4\pi\mu} \left( \cos^2\theta - \ln \frac{1}{2} + 1 \right)
\end{aligned}$$

Where  $l$  is the length of the boundary element. Thus we will have

$$G_{22} = \frac{1}{4\pi\mu} \left( \cos^2\theta - \ln \frac{1}{2} + 1 \right) \quad (40)$$

$$G_{12} = G_{21} = \frac{1}{4\pi\mu} \sin\theta \cos\theta \quad (41)$$

Also, regarding that  $r_k n_k = 0$ , then the integrals including  $K_{kj}$  will equal zero.

After calculating the above integrals with the Gauss method, the following equation system will be reached

$$Gt - Hu = (G\hat{T} - H\hat{U}) \quad (42)$$

In the relation above,  $\mathbf{G}$  and  $\mathbf{H}$  are square or rectangular matrices and their elements are gained by the integration of  $K_{kj}$  and  $G_{ik}$  cores on the boundary elements. When calculating for boundary points, the matrices are square and when calculating for the internal points, they will be rectangular. By placing the  $\alpha$  numbers from Equation (36), the following can be written

$$Gt - Hu = Su \quad (43)$$

Where

$$s = (G\hat{T} - H\hat{U})F^{-1}U_jF_jF^{-1} \quad (44)$$

All the above matrices – except  $U_j$  – are geometric functions and needs to be calculated.

### 3. NUMERICAL SOLUTION RESULTS

This section examines the results of solving the Navier-Stokes equations in a cavity for different Reynolds numbers. In order to solve the problem, constant elements are used, which on one hand, has caused them to be applied more easily in the computer program, and on the other hand, has solved the problem of the angles. In case of using linear elements or elements in higher levels, there will be nodes in the angles of the computational area. Regarding that the traction energies are not continual in the angles, having nodes in the angles will cause difficulties, since in that case the traction energies in the angles are assumed to be continual. One of the methods of solving this problem is using discontinuous elements, which will be proposed in future researches.

A comparison of the numeral solution results by the finite difference method and BEM confirms the accuracy of the solutions gained, as presented in Figures 4 and 5. By increasing the Reynolds number, the effect of the fluid inertia gradually increases and therefore, more information on the fluid behavior is required in the computational area in order to have more accuracy for the solutions found. Choosing more internal points will cause more information to be available in the computational domain and therefore, the more the Reynolds number increases, the more it is required for the number of internal points to increase [17].

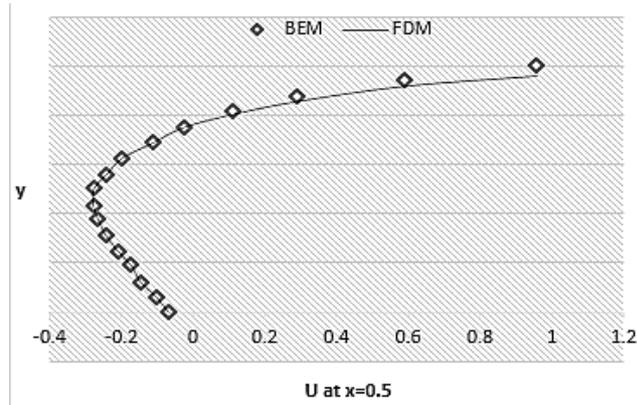


Figure 4: The Horizontal Fluid Velocity in the Cavity's Central Line in  $Re = 10$

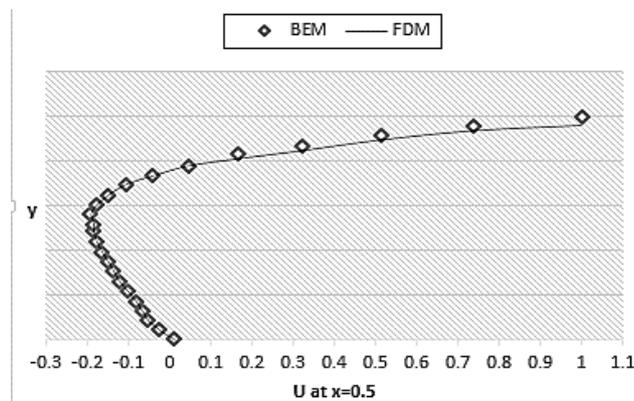


Figure 5: The Horizontal Fluid Velocity in the Cavity's Central Line in  $Re = 100$ .

40 boundary elements are used for  $Re = 10$ , and 96 boundary elements are considered for  $Re = 100$ . Also, 81 and 529 internal points are considered for  $Re = 10$  and  $Re = 100$  respectively.

#### 4. CONCLUSIONS

The BEM has been recently recognized as one of the very useful tools in analyzing many of the mechanical problems. Calculation field discretization is not needed and solving the problem using boundary information are the most important features and advantages of this method. Using this method to solve nonlinear problems and also linear problems dealing with integral terms on the field resulted from the main expression has gained more importance with proposing the Dual Reciprocity Method (DRM). This paper discussed how this method was applied for solving incompressible Navier-Stokes equations.

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